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Subpixel measurement of correlation algorithms based on Gaussian spot

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A theoretical study of general interpolation method is given, and different correlation algorithms based on Gaussian spot are compared. By using correlation algorithm combined with general interpolation, we can achieve a best interpolation method for different correlation algorithm. The best interpolation method for different correlation algorithms is achieved, meanwhile, we find that the best interpolation method is related to equivalent Gauss width of Gaussian spot under the ideal situations.

Keywords: *subpixel measurement, correlation algorithms, general interpolation method, Gaussian spot.*

OCIS codes: *100.0100, 100.3008.*

Субпиксельные измерения с использованием корреляционных алгоритмов на базе гауссовского распределения

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Проанализированы методы генеральной интерполяции, проведено сравнение корреляционных алгоритмов, использующих гауссовские процессы.

Используя корреляционный алгоритм в сочетании с генеральной интерполяцией, разработан наилучший среди корреляционных алгоритмов метод интерполяции, соответствующий в идеальных ситуациях эквивалентной ширине гауссовского распределения.

Ключевые слова: *субпиксельные измерения, корреляционный алгоритм, метод генеральной интерполяции, гауссовское распределение.*

Коды OCIS: *100.0100, 100.3008.*

1. INTRODUCTION

Correlation algorithms are widely applied in many fields [1–3], such as pattern recognition, target tracking, distance measurement and so on. As everyone knows, centroid algorithm is used to calculate the offset of each spot in the focal plane of Shack–Hartmann wavefront sensor, and the offset is subpixel. On the contrary, the correlation algorithm can't directly obtain the subpixel offset without the interpolation process, since different interpolation method could correspond to different subpixel precision.

There are many kinds of interpolation methods in practical application [4, 5]. To obtain the best interpolation method and achieve the subpixel measuring precision, the offsets obtained by different interpolation methods are compared.

Based on polynomial interpolation, like equiangular line fitting (ELF) [6] and parabola interpolation (PI) [7], we have put forward the general interpolation (GI) method, which can be regarded as the extension of polynomial interpolation method. We use GI method to achieve subpixel offset with three different correlation algorithms.

2. ELF AND PI METHOD

Before we go deep into the discussion of GI method, we could pay more attention to ELF method and PI method.

The equations of ELF and PI are shown below.

$$f_1(x) = a_1 |x - x_1| + b_1, \qquad (1)$$

$$f_2(x) = a_2 |x - x_2|^2 + b_2 = a'x^2 + b'x + c', \qquad (2)$$

where x_1 and x_2 denote the center point of Equation (1) and Equation (2) respectively, parameters a_1, b_1, a_2, b_2, a', b' and c' related to the sample points to estimate the center of each equations.

Given three sample points in the neighborhood of the peak (or valley) value are $(k-1, k1)$, $(k, k2)$ and $(k+1, k3)$, which we can safely remove k (integer) for the sake of simplicity. Then, we can use the two interpolation methods to calculate the center point of each equation as the sampling frequency makes x_1 or x_2 unknown.

Now, we can show two curves in Fig. 1 for estimate the center point with using the three sample points.

Then, we can calculate the abscissa of the center point of each method with using $(-1, k1)$, $(0, k2)$ and $(+1, k3)$, the final complete result is

$$x_1 = \begin{cases} \dfrac{1}{2}\dfrac{k3 - k1}{k2 - k1}, & k1 < k3 < k2 \\ \dfrac{1}{2}\dfrac{k3 - k1}{k2 - k3}, & k3 < k1 < k2 \end{cases}, \qquad (3)$$

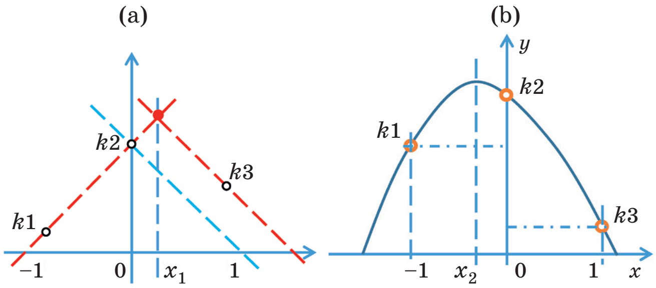

Fig. 1. Using three sample points to estimate the center point, ELF method (a) and PI method (b).

$$x_2 = -\frac{1}{2}\frac{k3 - k1}{k3 + k1 - 2\cdot k2} = x_{pa}. \qquad (4)$$

We also can give the relationship between parameters a', b', c' and $k1$, $k2$, $k3$ shown as follows:

$$\begin{cases} a' = (k1 + k3 - 2k2)/2, \\ b' = (-k1 + k3)/2, \\ c' = k3. \end{cases} \qquad (5)$$

3. GI METHOD

From part 2, we know the abscissa of the center point of ELF and PI could estimate when using three sample points in the neighborhood of the peak (or valley) value.

According to the equations of ELF and PI, we can easily give the equation of GI method as shown in Equation (6), where b and a are related to coordinates of the point to be interpolated, x_0 is the subpixel offset to be calculated and m represents the change of interpolation curves ($m > 0$).

$$f(x, m) = a|x - x_0|^m + b. \qquad (6)$$

This equation could simplify as ELF (Equation (1)) and PI (Equation (2)) when $m = 1$ and $m = 2$ respectively.

The same process of deduction as part 2, we can deduce the abscissa of the center point of GI method, the final result also related to the three points $(k-1, k1)$, $(k, k2)$ and $(k+1, k3)$, which means we can concern more about the value of three points instead of the exactly value of a and b.

We have three sample points in the neighborhood of the peak (or valley) value, the coordinate values are $(-1, k1)$, $(0, k2)$ and $(1, k3)$ respectively. $(0, k2)$ is the coordinate value of the peak (or valley) value. With the inequality $k2 > k3 > k1$ and the subpixel offset x_0 satisfies $0 < x_0 < 0.5$ when the image sampling rate is enough. Equation (7) is easily derived, where x_{pa} means the subpixel offset calculated by Equation (4).

$$4x_{pa}x_0^m + (1 - 2x_{pa})(1 + x_0)^m - \\ - (1 + 2x_{pa})(1 - x_0)^m = 0. \qquad (7)$$

In the same way, we have another situation in which satisfies $k2 > k1 > k3$ and $-0.5 < x_0 < 0$ with Equation (8).

$$4x_{pa}(-x_0)^m + (1 - 2x_{pa})(1 + x_0)^m - \\ -(1 + 2x_{pa})(1 - x_0)^m = 0. \qquad (8)$$

Although we can solve equation (7) or equation (8) with using professional mathematics software (Maple or Matlab), it's hard to use in practical application.

4. SUBPIXEL MEASUREMENT OF CORRELATION ALGORITHMS

According to the optical theory, the diffraction spot distribution of different shape aperture in the focal plane is different, for example, the diffraction spot formed by circular hole is Airy spot, and the diffraction spot formed by square hole could different. It is not convenient to use this distribution of diffraction spot to evaluate in the theoretical derivation and analysis. The most commonly used method is using the Gaussian spot instead of the actual spot for theoretical derivation. The distribution of Gaussian spot can be expressed as shown in Equation (9) where the amplitude values have been normalized. σ_x and σ_y represent equivalent Gauss width (EGW) of the Gaussian spot in x and y direction respectively. Considering a reference Gaussian spot only have offset (α, β) relative to $I(x, y)$, see Equation (10). For the convenience and simplicity of formulas′ deducing, if there is no special note, following derivation will only consider x direction.

$$I(x,y)=\exp\left(-\frac{x^2}{2\sigma_x^2}\right)\exp\left(-\frac{y^2}{2\sigma_y^2}\right), \quad (9)$$

$$I_r(x,y)=\exp\left[-\frac{(x-\alpha)^2}{2\sigma_x^2}\right]\exp\left[-\frac{(y-\beta)^2}{2\sigma_y^2}\right]. \quad (10)$$

There are so many correlation algorithms in practical application. Here we only consider three different cases: Absolute Difference Function (ADF), Absolute Difference Function-Squared (ADF2) and Cross-Correlation Coefficient (CCC). One-dimensional continuous form of correlation algorithms based on one-dimensional Gaussian distribution are shown as follows. σ and α represent EGW of the Gaussian distribution and relative offset respectively.

$$\mathrm{CCC}(x)=\int_{-\infty}^{\infty}\exp\left[-\frac{(u-\alpha)^2}{2\sigma^2}\right]\exp\left[-\frac{(u+x)^2}{2\sigma^2}\right]du, \quad (11)$$

$$\mathrm{ADF}(x)=\int_{-\infty}^{\infty}\left|\exp\left[-\frac{(u-\alpha)^2}{2\sigma^2}\right]-\exp\left[-\frac{(u+x)^2}{2\sigma^2}\right]\right|du, \quad (12)$$

$$\mathrm{ADF2}(x)=\left\{\int_{-\infty}^{\infty}\left|\exp\left[-\frac{(u-\alpha)^2}{2\sigma^2}\right]-\exp\left[-\frac{(u+x)^2}{2\sigma^2}\right]\right|du\right\}^2. \quad (13)$$

With using Maple 17, we can derive the result of each correlation algorithm as follows.

$$\mathrm{CCC}(x)=\sqrt{\pi}\sigma\exp\left[-\frac{(x+\alpha)^2}{4\sigma^2}\right], \quad (14)$$

$$\mathrm{ADF}(x)=2\sqrt{2\pi}\sigma\left|\mathrm{erf}\left(\frac{x+\alpha}{2\sqrt{2}\sigma}\right)\right|, \quad (15)$$

$$\mathrm{ADF2}(x)=8\pi\sigma^2\left[\mathrm{erf}\left(\frac{x+\alpha}{2\sqrt{2}\sigma}\right)\right]^2. \quad (16)$$

The erf(x) represents error function and satisfies $\mathrm{erf}(\infty)=1$, $\mathrm{erf}(-x)=-\mathrm{erf}(x)$.

$$\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x\exp(-t^2)dt. \quad (17)$$

For simplicity, assume the sampling interval is 1, the real subpixel offset x_{real} could range from −0.5 pixel to 0.5 pixel. The calculate offset has a minus sign relative to the real offset as the infinite interval integral of correlation algorithms. So the final calculate offsets of different correlation algorithms based on PI method are given below.

$$x_{\mathrm{CCC},pa}=\frac{1}{2}\frac{\mathrm{CCC}(1)-\mathrm{CCC}(-1)}{\mathrm{CCC}(1)+\mathrm{CCC}(-1)-2\mathrm{CCC}(0)}, \quad (18)$$

$$x_{\mathrm{ADF},pa}=\frac{1}{2}\frac{\mathrm{ADF}(1)-\mathrm{ADF}(-1)}{\mathrm{ADF}(1)+\mathrm{ADF}(-1)-2\mathrm{ADF}(0)}, \quad (19)$$

$$x_{\mathrm{ADF2},pa}=\frac{1}{2}\frac{\mathrm{ADF2}(1)-\mathrm{ADF2}(-1)}{\mathrm{ADF2}(1)+\mathrm{ADF2}(-1)-2\mathrm{ADF2}(0)}. \quad (20)$$

Substituting Equations (14)–(16) into Equations (18)–(20) respectively, and then substituting the result into Equation (7) or (8) according to the real subpixel offset have been defined above with different m values, finally we can derive the estimated subpixel offset x_{cal}.

It′s not easy to compare all the results as too many variables, let′s define a new quantity called error of mean square error (EMSE) and expressed as Std_x, where N is the number of real offset.

$$\mathrm{Std}_x=\sqrt{\sum_{i=1}^{N}\left[x_{\mathrm{real}}(i)-x_{\mathrm{cal}}(i)\right]^2}. \quad (21)$$

If we choose $\sigma = 1$ pixel, real subpixel offset ranges from 0 pixel to 0.5 pixel with step length 0.05 pixel and m ranges from 0.1 to 3 with step length 0.05. The results of different correlation algorithms are shown as Fig. 2. From Fig. 2 we know, each correlation algorithm has a best interpolation curve when conditions are determined.

Obviously, the distribution of Std_x will changing with different EGW. Let′s explore what′s the relationship between EGW and best value m_{opt}. First, we choose EGW ranges from 0.5 pixel to 4 pixels with

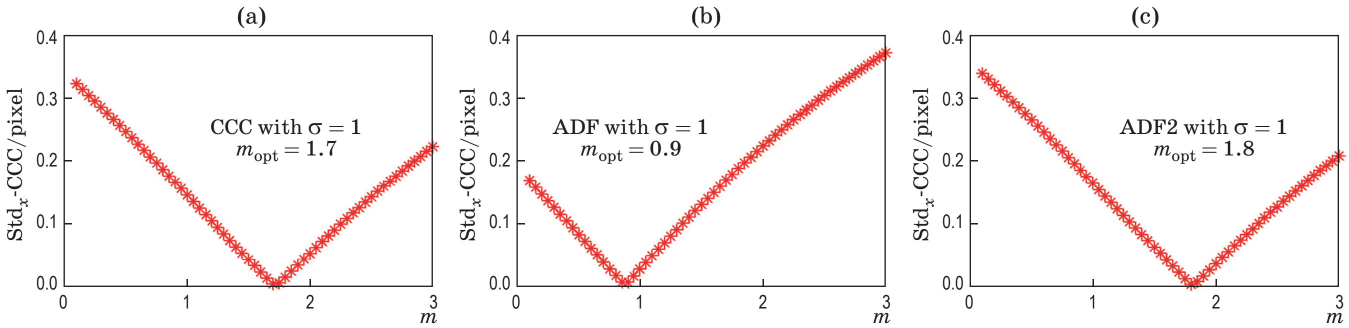

Fig. 2. The distribution of Std_x with m ranges from 0.1 to 3 and $\sigma = 1$ pixels, m_{opt} gives the best interpolation curve with different algorithms.

step length 0.02 pixel (0.5–1.5 pixel) and 0.2 pixel (1.6–4 pixels) as shown in Fig. 3, real subpixel offset x_{real} ranges from 0 pixel to 0.5 pixel with step length 0.05 pixel and m ranges from 0.1 to 3 with step length 0.05. The final results are shown in Fig. 4.

From Fig. 4, it becomes clear that the best value m increases first and then remains almost unchanged with increasing of the EGW for all the correlation algorithms. The best value changes with EGW from Fig. 5 is shown below. The result indicates when the EGW increases to greater than or equal to about 3 pixels, the best interpolation method of CCC and ADF2 is PI method (where $m = 2$), the best interpolation method of ADF is ELF method (where $m = 1$). This means that no matter which correlation algorithm someone chooses, it can obtain very good subpixel interpolation accuracy as long as the best interpolation curve is selected.

In the discussion above about correlation function, the ideal formula of correlation algorithms has been deduced. In a real-world scenario, more factors need to be considered, like discrete sampling of the system, truncation effect of lens, different size of reference image and real image and so on. As this is not the focus of this article, we only give the final form of correlation function with considering truncation effect, discrete sampling and different size of reference image and real image.

$$\mathrm{CCC}_{\mathrm{real}}(k) = \sqrt{\frac{\pi}{2}}\sigma \sum_{i=-\frac{L}{2}+k-1}^{-\frac{L}{2}+L_r+k-1} |A1(i)A2(i)|, \quad (22)$$

$$\mathrm{ADF}_{\mathrm{real}}(k) = \sqrt{\frac{\pi}{2}}\sigma \sum_{i=-\frac{L}{2}+k-1}^{-\frac{L}{2}+L_r+k-1} |A1(i) - A2(i)|, \quad (23)$$

$$\mathrm{ADF2}_{\mathrm{real}}(k) = \frac{\pi}{2}\sigma^2 \left[\sum_{i=-\frac{L}{2}+k-1}^{-\frac{L}{2}+L_r+k-1} |A1(i) - A2(i)|\right]^2, \quad (24)$$

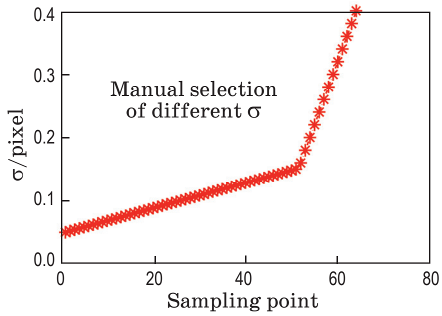

Fig. 3. Selecting 64 points in (0.5, 4) as the σ.

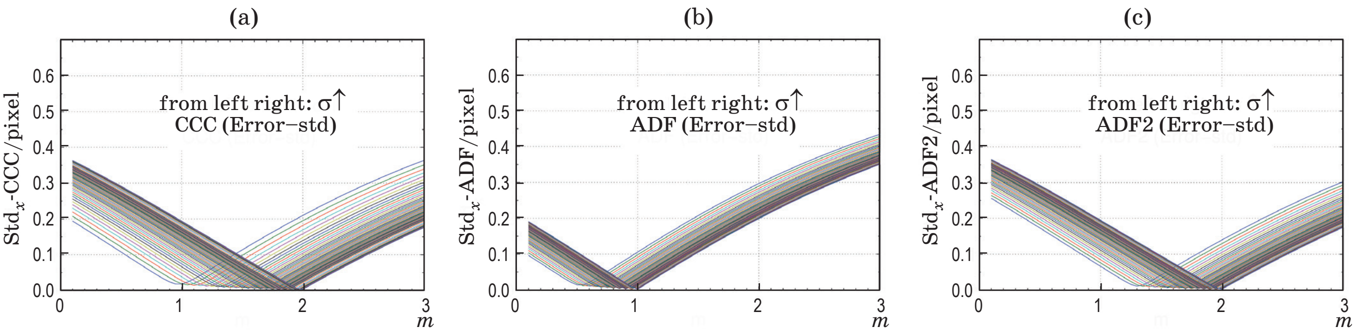

Fig. 4. The distribution of Std_x with m ranges from 0.1 to 3 and σ ranges from 0.5 to 4 using different correlation algorithms.

$$A1(i)=\mathrm{erf}\left(\frac{2\alpha+a-2i+1}{2\sqrt{2}\sigma}\right)+\mathrm{erf}\left(\frac{-2\alpha+a+2i-1}{2\sqrt{2}\sigma}\right), \quad (25)$$

$$A2(i)=\mathrm{erf}\left[\frac{a-2\left(i-\frac{L_r}{2}+\frac{L}{2}-k+1\right)+1}{2\sqrt{2}\sigma}\right]+ +\mathrm{erf}\left[\frac{a+2\left(i-\frac{L_r}{2}+\frac{L}{2}-k+1\right)-1}{2\sqrt{2}\sigma}\right]. \quad (26)$$

The meanings of these parameters from Equations (22)–(26) are given as follow: k is the discrete variables of correlation function, L – the size of $I(x)$, L_r – the size of $I_r(x)$ ($L_r < L$), σ – the EGW of $I(x)$ and $I_r(x)$, α – the real offset, a – the pixel size of CCD and i – the discrete pixel coordinates of $I(x)$ and $I_r(x)$. We can calculate the estimate offset in the same way above with these correlation functions by GI method. The results are more complicated as the real conditions must be considered, but the same conclusion could be obtained as each correlation algorithm can achieve high subpixel interpolation accuracy as long as the best interpolation curve is determined. We will not repeat them here anymore.

5. APPROXIMATE EXPRESSION OF GI METHOD

For the purpose of using GI method in actual hardware system (only consider the ideal case), we could give the approximate expression of GI method based on Taylor expansions (detailed derivation has been ignored). Consider the situation with calculating Equation (7) that the estimated subpixel satisfies $0 < x_0 < 0.5$ and inequality $0 < k1 < k3 < k2$, if takes first two orders ($n = 2$) approximation of Taylor expansions, then

$$x_0=\frac{3x_{pa}-mx_{pa}}{4x_{pa}-2mx_{pa}+1}, 0<m<3. \quad (27)$$

The full approximation (first two orders) calculation formula of GI method is shown below.

$$x_0=\begin{cases}\dfrac{3x_{pa}-mx_{pa}}{4x_{pa}-2mx_{pa}+1}, k1<k3<k2(0<m<3)\\ \dfrac{3x_{pa}-mx_{pa}}{-4x_{pa}+2mx_{pa}+1}, k3<k1<k2(0<m<3)\end{cases}. \quad (28)$$

At the same time, if we consider the result of CCC as shown in Fig. 4, the approximate result can be calculated by Equation (28). From Fig. 6, we know that the best interpolation method of CCC is a little bit different from Fig. 4, which means the approximation formula of GI method could underestimate or overestimate the best value m_{opt}. However, this approximation value of m also can give a high subpixel accuracy relative to PI or ELF when the EGW of Gaussian spot is relatively small (about 0.5 to 3 pixels for each correlation algorithm). With the increasing of EGW

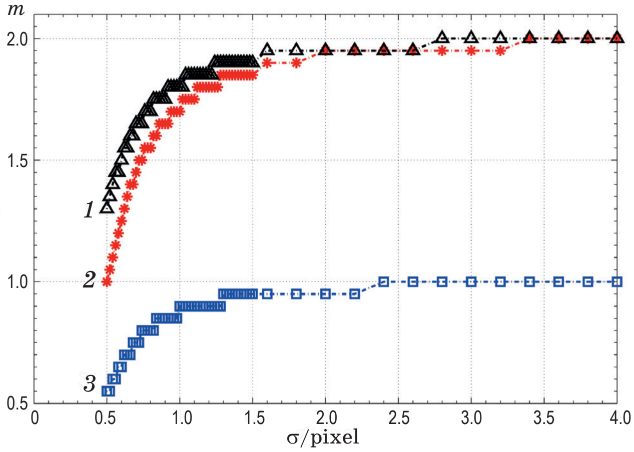

Fig. 5. The best values of m with different σ using different correlation algorithms from Fig. 4. The curve *1* represents CCC, the curve 2 represents ADF, and the curve 3 represents ADF2.

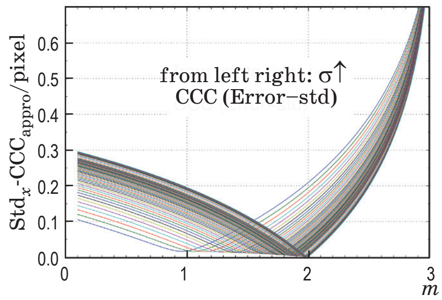

Fig. 6. The distribution of Std_x with m ranges from 0.1 to 3 and σ ranges from 0.5 to 4 using CCC based on approximation formula of GI method.

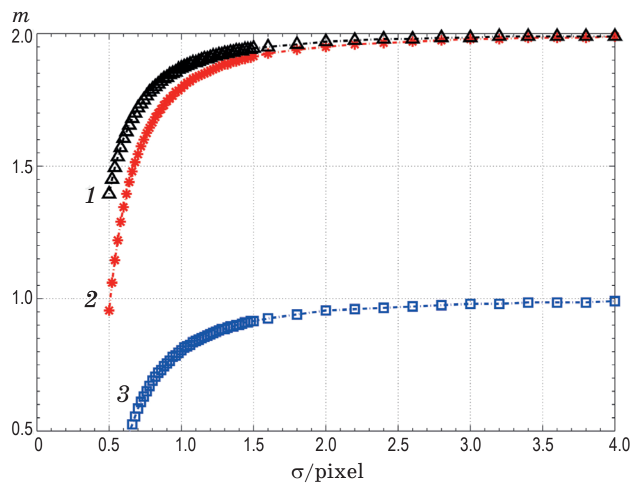

Fig. 7. The best values of m with different σ using approximation formula of GI method. The curve 1 represents CCC, the curve 2 represents ADF, and the curve 3 represents ADF2.

of Gaussian spot, the best interpolation method of CCC or ADF2 is also PI method, so that of ADF is also ELF method, which would be shown in Fig. 7.

6. DISCUSSION AND CONCLUSIONS

This paper introduces an extended interpolation method based on polynomial interpolation, we call GI method. PI and ELF methods are the special case of GI method when $m = 2$ and $m = 1$ respectively. With a detailed theoretical derivation of GI method, we have derived a nonlinear equation to achieve the subpixel offset. It′s easy to obtain the subpixel offset with mathematical software, such as Matlab or Maple.

With the Gaussian spot, the expression of three different correlation algorithms was obtained under ideal conditions. At the same time, the results of GI method with these correlation algorithms using EMSE have discussed, the results indicate that the best interpolation method of different correlation algorithms based on Gaussian spot is related to Equivalent Gauss width, and when the EGW increases to greater than or equal to about 3 pixels, the best interpolation method of CCC and ADF2 is PI ($m = 2$), the best interpolation method of ADF is ELF ($m = 1$). Under this situation, when the EGW is smaller than about 3 pixels, someone could always find an appropriate m as the best interpolation method to derive subpixel offset. The expression of correlation algorithms with considering discrete sampling of the system, truncation effect of lens, different size of reference image and real image are also given, which would closer to the actual situation, but we didn′t discuss the result based on GI method.

At the same time, the simplified expression of the nonlinear equation has given using Taylor expansion. It′s easy to realize in hardware system with this more simple calculation formula, but would introduce some deviation to the final result when the EGW of Gaussian spot is relative small (0.5 pixel to 3 pixels), and could also give the same result to the exact result when the EGW is greater than about 3 pixels.

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